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MATHEMATICAL MODEL OF AEROELASTIC VIBRATIONS OF CASCADES OF AXIAL TURBOMACHINE BLADES, CAUSED BY CIRCUMFERENTIAL NONUNIFORMITY OF THE FLOW

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A system of linear differential equations with time-dependent coefficients, which describes aeroelastic vibrations of blade cascades in a nonuniform flow, is derived. With the use of the model of an ideal incompressible fluid and the hypothesis of cylindrical sections, determination of aerodynamic forces acting on the blades is reduced to solving problems by methods fairly well developed in the theory of cascades in unsteady flow. The possibility of the emergence of a parametric resonance is analyzed. It is demonstrated that circumferential nonuniformity of the flow in the turbomachine duct can substantially reduce the critical velocity of the cascade flutter.

Key words: *cascade of blades, aeroelastic vibrations, flutter, flow nonuniformity, parametric resonance.*

Introduction. An inherent feature of the velocity field in the duct of axial turbomachines is its circumferential nonuniformity. It appears because the flow is perturbed by various turbomachine elements, for instance, rotor wheels or guide and straightener blades. When the working wheel rotates in a circumferentially nonuniform flow, its blades experience the action of periodic unsteady forces exciting blade vibrations. If the working medium is a gas, then the unsteady aerodynamic forces acting on the wheel blades are sufficiently small, as compared with elastic and inertial forces induced in the case of blade vibrations. Therefore, the general problem of aeroelastic vibrations of a blade cascade decomposes in the linear approximation into three subproblems: 1) inherent vibrations of the cascade of blades in vacuum; 2) determining the unsteady aerodynamic characteristics of the cascade corresponding to its own modes in vacuum; 3) vibrations of the cascade with allowance for aerodynamic interaction of the blades.

The first subproblem has been treated in much detail [1, 2]. In studying the problems of aeroelastic vibrations of cascades in a nonuniform flow, the second subproblem has been adequately solved only for a plane model of an unsteady flow through the cascade [3–6].

Many publications deal with solving the third subproblem within the model of aeroelastic vibrations of cascades, which are described by linear differential equations with constant coefficients [3, 4, 7–9]. In reality, the coefficients of the aerodynamic forces in the corresponding differential equations are time-dependent because of circumferential nonuniformity of the flow in the turbomachine duct. The presence of such forces can lead to the emergence of a parametric resonance; the probability of this resonance in the flow through cascades was analyzed previously in [10–12]. Conditions responsible for the emergence of this phenomenon, however, were actually not considered in those papers.

A system of differential equations with time-dependent coefficients that describe aeroelastic vibrations of axial turbomachine cascades, which are induced by circumferential nonuniformity of the flow, is derived in the present work. A method of determining the unsteady aerodynamic forces acting on the blades is developed, and the possibility of the emergence of a parametric resonance is analyzed.

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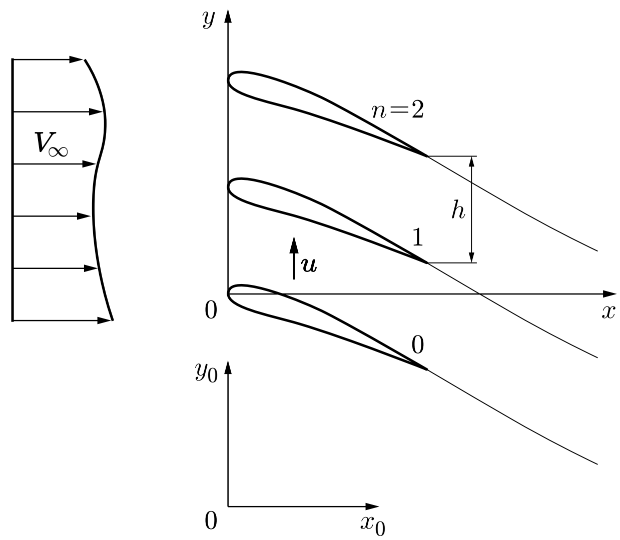

Fig. 1. Straight cascade of blades.

1. Formulation of the Problem. Let us consider the vibrations of an axial turbomachine cascade rotating in a incompressible fluid flow with a nonuniform circular velocity. Small geometric irregularity of the cascade, caused by technological inaccuracy of blade manufacturing and wheel assembling, is admitted in the general case, because the associated inhomogeneity of dynamic characteristics of the blades exerts a significant effect on their aeroelastic vibrations [7–9]. We assume that the effect of this inhomogeneity on the unsteady component of aerodynamic forces can be neglected. Therefore, we assume that the eigenfrequencies and eigenshape of blade vibrations in vacuum satisfy the condition

$$|\omega_{sn} - \omega_{s0}| \ll \omega_{00}, \qquad \psi_{sn} = \psi_s$$
$$(n = 0, 1, 2, \ldots, N_0 - 1, \quad s = 1, 2, 3, \ldots), \tag{1.1}$$

where ω_{sn} and ψ_s are the eigenfrequency and the eigenshape of the nth blade, which correspond to the sth mode of vibrations and are assumed to be known.

As the friction forces of the gas do not exert any significant effect on blade vibrations, the unsteady aerodynamic forces acting on the blades are sought within the framework of the ideal fluid model, similar to the commonly used procedure in problems of aeroelasticity of the airfoils and turbomachine cascades. In the case considered, we determine these forces on the basis of the hypothesis of cylindrical sections, which implies that the radial fluid flow in the axial turbomachine duct has an insignificant effect on the integral characteristics of aerodynamic interaction of cascade blades with the flow. According to this hypothesis, determining these characteristics reduces to solving problems of the flow through straight cascades, which are unfolded patterns of the cylindrical sections of the axial cascade.

Let us consider the flow through a straight cascade of blades, which is an unfolded pattern of the cylindrical section of radius z of the axial turbomachine cascade rotating with an angular velocity Ω in a circumferentially nonuniform fluid flow (see Fig. 1). In a motionless coordinate system (x_0, y_0), in which the cascade considered moves in the direction along the y_0 axis with a velocity $u = \Omega z$, the vector of the incoming flow velocity can be presented as

$$\boldsymbol{V}_\infty(x_0, y_0) = \boldsymbol{V}_0 + \varepsilon_1 \boldsymbol{V}_1(x_0, y_0), \qquad \boldsymbol{V}_0 = \text{const},$$
$$\max |\boldsymbol{V}_1| = |\boldsymbol{V}_0|, \qquad \varepsilon_1 \ll 1, \qquad \boldsymbol{V}_1(x_0, y_0) = \boldsymbol{V}_1(x_0, y_0 + H), \tag{1.2}$$

where $\boldsymbol{V}_1$ is the vector-function determining the nonuniformity of the incoming flow, $H = Nh$ is the period of this function, h is the cascade step, and N is a natural number to which the number of blades in the cascade N_0 is multiple. Let us present the periodic vector-function $\boldsymbol{V}_1(x_0, y_0)$ in the form of the Fourier series

$$\boldsymbol{V}_1(x_0, y_0) = \frac{1}{2}\,\boldsymbol{a}_0 + \sum_{r=1}^{\infty}\Big[\boldsymbol{a}_r(x_0)\cos\Big(\frac{2\pi r y_0}{H}\Big) + \boldsymbol{b}_r(x_0)\sin\Big(\frac{2\pi r y_0}{H}\Big)\Big] \tag{1.3}$$

and introduce a moving coordinate system (x, y) rigorously fitted to the cascade:

$$x = x_0, \qquad y = y_0 - ut. \tag{1.4}$$

Taking into account Eqs. (1.2)–(1.4), we can simplify the problem of the fluid flow through the cascade by presenting the relative free-stream velocity in the complex form

$$\boldsymbol{W}_\infty(x, y, t) = \boldsymbol{W}_0 + \boldsymbol{W}_1(x, y, t); \tag{1.5}$$

$$\boldsymbol{W}_1(x, y, t) = \varepsilon_1|\boldsymbol{W}_0|\sum_{r=1}^{\infty}[\boldsymbol{W}_{1r}\exp{(j\omega_r t)} + \bar{\boldsymbol{W}}_{1r}\exp{(-j\omega_r t)}], \tag{1.6}$$

where

$$\boldsymbol{W}_{1r} = \frac{1}{2|\boldsymbol{W}_0|}\,[\boldsymbol{a}_r(x) - j\boldsymbol{b}_r(x)]\exp\Big(j\,\frac{2\pi r y}{H}\Big),$$

$\bar{\boldsymbol{W}}_{1r}$ is the function that is complex conjugate to $\boldsymbol{W}_{1r}$, $\omega_r = \omega r$, $\omega = 2\pi u/H$, $\boldsymbol{W}_0 = \boldsymbol{V}_0 - \boldsymbol{u} = \text{const}$, and j is the imaginary time unit.

It follows from Eqs. (1.5) and (1.6) that the relative velocity of the flow incoming onto the cascade contains an unsteady component periodic in time. Therefore, the flow perturbation arising as a result of flow interaction with the cascade also contains an unsteady component in the relative coordinate system. Under the action of aerodynamic forces generated by this perturbation, the cascade blades perform forced vibrations, which, in turn, generate additional flow perturbations depending on the character of blade vibrations. The dependence of the coefficients of the corresponding aerodynamic forces in the generalized coordinates, which determine the law of blade vibrations, is periodic in time. Thus, when the cascade is exposed to a flow whose velocity contains a periodic component, the cascade blades perform not only forced, but also parametric vibrations. The task is to determine the unsteady aerodynamic forces acting on the cascade blades, which are induced by periodic nonuniformity of the incoming flow, and to evaluate the influence of this nonuniformity on the character of blade vibrations, in particular, on their stability.

2. Determining the Perturbed Component of Flow Velocity. The relative flow velocity in the vicinity of the cascade can be presented as a sum of four terms:

$$\boldsymbol{W}(x, y, t) = \boldsymbol{W}_\infty(x, y, t) + \boldsymbol{w}_0(x, y) + \boldsymbol{w}_1(x, y, t) + \boldsymbol{w}(x, y, t).$$

Here, $\boldsymbol{W}_\infty$ is the function defined in the form (1.5), (1.6), $\boldsymbol{w}_0$ is the steady component of the flow velocity perturbation, which is assumed to be known, and $\boldsymbol{w}_1$ and $\boldsymbol{w}$ are the unsteady components of the perturbed velocity of the fluid, which are induced by interaction of the cascade of motionless blades with the nonuniform flow and by blade vibrations, respectively.

By virtue of the assumptions made in formulating the problem, we confine ourselves to determining the unsteady components of the fluid flow velocity perturbation in the linear approximation and within the framework of the model of an ideal incompressible fluid. Note that the order of magnitude of the corresponding components of this perturbation and their time evolution are determined by the boundary conditions of non-penetration of the incoming flow through the cascade blades. For convenience of transformations, all linear sizes in what follows are assumed to be normalized to the mean chord of the cascade blades b.

Following the principle of superposition and taking into account Eq. (1.6), we present the velocity potential of the unsteady component of the fluid flow, which is induced by nonuniformity of the flow incoming onto the motionless blades, in the following form:

$$\varphi_1 = \varepsilon_1|\boldsymbol{W}_0|b\sum_{r=1}^{\infty}[\varphi_{1r}(x, y)\exp{(j\omega_r t)} + \bar{\varphi}_{1r}(x, y)\exp{(-j\omega_r t)}]. \tag{2.1}$$

Here, φ_{1r} is the amplitude function of the velocity potential of fluid vibrations with a frequency ω_r and $\bar{\varphi}_{1r}$ is the function that is complex conjugate to the function φ_{1r}. According to Eq. (1.6), the function φ_{1r} should satisfy the condition

$$\frac{\partial \varphi_{1r}(x, y+nh)}{\partial \nu_n} = -W_{1r\nu_0}(x, y) \exp(j\mu_r n), \qquad (x, y) \in L_0$$

$$(n = 0, 1, 2, \ldots, N_0 - 1),$$

where ν_n is the direction of the normal to the blade contour L_n.

Therefore, the conditions for the neighboring blades differ only by a constant phase shift $\mu_r = 2\pi r/N$. This circumstance allows us to apply well-developed methods in the cascade theory for unsteady flow [3–6] to determine the sought function φ_{1r}. Thus, this function possesses a property of generalized periodicity of the form

$$\varphi_{1r}(x, y+nh) = \varphi_{1r}(x, y) \exp(j\mu_r n), \qquad (x, y) \in L_0. \tag{2.2}$$

According to Eq. (2.1), the function $\boldsymbol{w}_1$, which is determined via φ_1, has the form

$$\boldsymbol{w}_1(x, y, t) = \varepsilon_1 |\boldsymbol{W}_0| \sum_{r=1}^{\infty} [\boldsymbol{w}_{1r}(x, y) \exp(j\omega_r t) + \bar{\boldsymbol{w}}_{1r}(x, y) \exp(-j\omega_r t)]. \tag{2.3}$$

To determine flow perturbations induced by blade vibrations, we present the general law of their vibrations, with allowance for Eq. (1.1), in the complex form as

$$z_n = \sum_{r=1}^{\infty} \sum_{s=1}^{N_1} x_{nrs} \psi_s(x, y) \exp(j\omega_r t) \qquad (n = 1, 2, \ldots, N_0), \tag{2.4}$$

where x_{nrs} is the dimensionless value of the generalized coordinate determining the level of vibrations of the nth blade with respect to the rth harmonic and the sth mode; N_1 is the number of the degrees of freedom of blade vibrations taken into account. In this case, the potential of the perturbed component of the fluid velocity can be presented in the following form with allowance for Eq. (2.4):

$$\varphi(x, y) = |\boldsymbol{W}_0| b \sum_{r=1}^{\infty} \sum_{s=1}^{N_1} \sum_{n=0}^{N_0-1} x_{nrs} \varphi_{nrs}(x, y) \exp(j\omega_r t). \tag{2.5}$$

Here, φ_{nrs} is the dimensionless value of the amplitude function of the unsteady component of the velocity potential, which arises due to vibrations of the nth blade with respect to the rth harmonic and the sth mode with a unit amplitude. The function φ_{nrs} is determined by solving the boundary-value problem with the blade contours subjected to conditions of the form

$$\begin{gathered}\frac{\partial \varphi_{nrs}}{\partial \nu_n} = \Big[jk_r \psi_s(x, y) + \frac{\partial}{\partial \sigma}\Big(\frac{|\boldsymbol{W}_0 + \boldsymbol{w}_0|}{|\boldsymbol{W}_0|}\, \psi_s(x, y)\Big)\Big], \qquad (x, y) \in L_n, \\ \frac{\partial \varphi_{mrs}}{\partial \nu_m} = 0, \qquad (x, y) \in L_m, \quad m \neq n,\end{gathered} \tag{2.6}$$

where $k_r = \omega_r b/|\boldsymbol{W}_0|$. It should be noted that it is rather difficult to solve problems with the boundary condition in the form (2.6). Let us demonstrate that determining the sought functions φ_{nrs} can be reduced to solving simpler problems of the fluid perturbation induced by simultaneous vibrations of the cascade of blades with identical amplitudes and identical shifts of the phases of vibrations of the neighboring blades. For this purpose, we introduce the functions $\tilde{\varphi}_{lrs}$ possessing the property of generalized periodicity

$$\tilde{\varphi}_{lrs}(x, y+mh) = \tilde{\varphi}_{lrs}(x, y) \exp\Big(j\, \frac{2\pi ml}{N_0}\Big)$$

$$(m = 1, 2, \ldots, N_0 - 1, \quad l = 1, 2, \ldots, N_0 - 1),$$

which are the solutions of the corresponding boundary-value problems under the condition

$$\frac{\partial \tilde{\varphi}_{lrs}(x, y+mh)}{\partial \nu_m} = \Big[jk_r \psi_s(x, y) + \frac{\partial}{\partial \sigma}\Big(\frac{|\boldsymbol{W}_0 + \boldsymbol{w}_0|}{|\boldsymbol{W}_0|}\, \psi_s(x, y)\Big)\Big] \exp\Big(j\, \frac{2\pi ml}{N_0}\Big),$$

$$(x, y) \in L_0 \qquad (m = 0, 1, 2, \ldots, N_0 - 1).$$

Then, the functions φ_{nrs} can be presented as

$$\varphi_{nrs} = \frac{1}{N_0} \sum_{l=0}^{N_0-1} \exp\Big(-j\,\frac{2\pi nl}{N_0}\Big)\tilde{\varphi}_{lrs},$$

because the expressions for their derivatives along the normal to the blade contours

$$\frac{\partial \varphi_{nrs}}{\partial \nu_m} = \frac{1}{N_0} \sum_{l=0}^{N_0-1} \exp\Big(-j\,\frac{2\pi nl}{N_0}\Big)\frac{\partial \tilde{\varphi}_{lrs}}{\partial \nu_m},$$

$$(x, y) \in L_0 \qquad (m = 0, 1, 2, \ldots, N_0 - 1)$$

are equivalent to the non-penetration conditions (2.6).

With allowance for Eq. (2.5), the perturbed velocity component induced by blade vibrations can be presented as

$$\boldsymbol{w}(x, y, t) = |\boldsymbol{W}_0| \sum_{n=0}^{N_0-1} \sum_{r=1}^{\infty} \sum_{s=1}^{N_1} x_{nrs} \boldsymbol{w}_{nrs}(x, y) \exp{(j\omega_r t)}. \tag{2.7}$$

3. Determining the Unsteady Component of Aerodynamic Forces. To determine the unsteady aerodynamic forces acting on the cascade blades, it is necessary to know the unsteady component of static pressure of the fluid on the blade surfaces. Within the framework of the hypothesis of cylindrical sections, the static pressure on the blade surfaces $(x, y) \in L_n$ is obtained from the Lamb–Gromeka equation in the form

$$p_0 = -\rho\Big(\frac{\partial \varphi_0}{\partial t} + \boldsymbol{W}_0(\boldsymbol{W}_1 + \boldsymbol{w}_1) + \boldsymbol{W}_0\boldsymbol{w} + (\boldsymbol{W}_1 + \boldsymbol{w}_1)\boldsymbol{w} + \frac{1}{2}\,(\boldsymbol{W}_1 + \boldsymbol{w}_1)^2 + \frac{1}{2}\,\boldsymbol{w}^2\Big),$$

where ρ is the fluid density. The component of this pressure equal to

$$p_1(x, y, z) = -\rho\Big(\frac{\partial \varphi_1}{\partial t} + \boldsymbol{W}_0(\boldsymbol{W}_1 + \boldsymbol{w}_1) + \frac{1}{2}\,(\boldsymbol{W}_1 + \boldsymbol{w}_1)^2\Big) \tag{3.1}$$

is independent of blade vibrations and determines the aerodynamic forces per unit length of the blade in the cylindrical section of the cascade of radius z, which excite forced vibrations of the blades. According to Eqs. (1.6) and (2.3), the last term in the right side of Eq. (3.1) is a quantity of the second order of smallness; hence, this term is neglected in further considerations.

The expression for static pressure caused by blade vibrations has the form

$$p(x, y, z) = -\rho\Big(\frac{\partial \varphi}{\partial t} + \boldsymbol{W}_0\boldsymbol{w} + (\boldsymbol{W}_1 + \boldsymbol{w}_1)\boldsymbol{w} + \frac{1}{2}\,\boldsymbol{w}^2\Big). \tag{3.2}$$

It will be demonstrated below that the quantity $\boldsymbol{w}$ has a higher order of smallness than $\boldsymbol{W}_1$ in the regimes considered; therefore, the last term in the right side of Eq. (3.2) can also be neglected.

The second term of Eq. (3.2) is the product of two complex functions. The real process is described only by their real parts. Therefore, to determine the aerodynamic forces acting on the blades, we have to pass to the real form of their expressions. It also seems reasonable to present the generalized coordinates that describe steady vibrations of the blades in the real form as well:

$$y_{nrs} = \mathrm{Re}\,[x_{nrs} \exp{(j\omega_r t)}] = \alpha_{nrs} \cos{(\omega_r t + \beta_{nrs})}. \tag{3.3}$$

Here, $\alpha_{nrs} = |x_{nrs}|$ and $\beta_{nrs} = \arctan\,[\mathrm{Im}\,(x_{nrs})/\,\mathrm{Re}\,(x_{nrs})]$ are the amplitudes and phases of vibrations. It should be noted that $\mathrm{Im}\,[x_{nrs} \exp{(j\omega_r t)}] = -\dot{y}_{nrs}/\omega_r$.

Taking into account Eqs. (2.1)–(2.3) and (3.1), we can present the real parts of the generalized aerodynamic forces exciting forced vibrations of the nth blade in the cascade in the vector form:

$$\boldsymbol{Q}_{0n} = -\varepsilon_1 q b^2 \boldsymbol{F}_n,$$
$$\boldsymbol{F}_n = \sum_{r=1}^{\infty} \Big[\mathrm{Re}\,(\boldsymbol{F}_{0r}\,\mathrm{e}^{j\mu_r n}) \cos{(\omega_r t)} - \mathrm{Im}\,(\boldsymbol{F}_{0r}\,\mathrm{e}^{j\mu_r n}) \sin{(\omega_r t)}\Big]. \tag{3.4}$$

Here, $q = \rho W_0^2/2$ is the dynamic pressure of the incoming flow and $\boldsymbol{F}_{0r}$ is the vector whose components f_s^{0r} determine (in the complex form) the aerodynamic forces exciting vibrations of the initial blade ($n = 0$) with the frequencies ω_r and sth modes:

$$f_s^{0r} = 2 \int\limits_{R_0}^{R_{end}} \oint\limits_{L_0(z)} \Big(jk_r \varphi_{1r} + \frac{\boldsymbol{W}_0 + \boldsymbol{w}_0}{|\boldsymbol{W}_0|} (\boldsymbol{W}_{1r} + \boldsymbol{w}_{1r}) \Big) \psi_s \, d\sigma \, dz$$

(R_0 and R_{end} are the radii of the cylindrical sections of the blade normalized to b and determining the initial and final points of the blades).

The generalized aerodynamic forces arising on the nth blade because of cascade vibrations and exciting blade vibrations of the sth mode are determined in the real form with allowance for Eq. (3.2) by the formula

$$Q_{ns} = -\rho b^2 \sum_{s=1}^{N_1} \int\limits_{R_0}^{R_{end}} \int\limits_{L_n(z)} \Big[\mathrm{Re} \Big(\frac{\partial \varphi}{\partial t} + \boldsymbol{W}_0 \boldsymbol{w} \Big) + \varepsilon_1 \, \mathrm{Re} \, (\boldsymbol{W}_1 + \boldsymbol{w}_1) \, \mathrm{Re} \, \boldsymbol{w} \Big] \psi_s \, d\sigma \, dz. \tag{3.5}$$

Substituting Eqs. (1.6), (2.3), (2.5), (2.7) into Eq. (3.5), we obtain an expression, which can be presented in the matrix form as

$$\boldsymbol{Q}_n = -qb^2 \sum_{m=0}^{N_0-1} \sum_{r=1}^{\infty} \Big[\Big(A_{nmr} Y_{mr} + \frac{1}{\omega_r} B_{nmr} \dot{\boldsymbol{Y}}_{mr} \Big) + \varepsilon_1 \sum_{l=1}^{N_2} \Big(C_{nmrl}(t) \boldsymbol{Y}_{mr} + \frac{1}{\omega_r} D_{nmrl}(t) \dot{\boldsymbol{Y}}_{mr} \Big) \Big] \tag{3.6}$$

($\boldsymbol{Q}_n$ and $\boldsymbol{Y}_{mr}$ are the vectors with the components $q_s^n = Q_{ns}$ and $y_u^{mr} = y_{mru}$). The elements of the matrices of Eq. (3.6) with the appropriate notation are determined as follows:

$$a_{su}^{nmr} = 2 \int\limits_{R_0}^{R_{end}} \oint\limits_{L_n(z)} \mathrm{Re} \, (jk_r \varphi_{mur} + w_{mur}) \psi_s \, d\sigma \, dz,$$

$$b_{su}^{nmr} = 2 \int\limits_{R_0}^{R_{end}} \oint\limits_{L_n(z)} \mathrm{Im} \, (jk_r \varphi_{mur} + w_{mur}) \psi_s \, d\sigma \, dz,$$

$$c_{su}^{nml} = 2 \int\limits_{R_0}^{R_{end}} \oint\limits_{L_n(z)} W_l(t) \, \mathrm{Re} \, (w_{mur}) \psi_s \, d\sigma \, dz,$$

$$d_{su}^{nml} = 2 \int\limits_{R_0}^{R_{end}} \oint\limits_{L_n(z)} W_l(t) \, \mathrm{Im} \, (w_{mur}) \psi_s \, d\sigma \, dz.$$

Here, $W_l = 2 \, \mathrm{Re} \, [(\boldsymbol{W}_{1l} + \boldsymbol{w}_{1l}) \exp{(j\omega_l t)}]$.

4. Differential Equations of Blade Vibrations Caused by Periodic Nonuniformity of the Incoming Flow. Within the framework of the considered model of aerodynamic interaction between the cascade and the fluid, we write the system of differential equations of blade vibrations. In accordance with [13] and Eqs. (3.4) and (3.6), this system can be presented as

$$\mathrm{diag} \, (M_s) \ddot{\boldsymbol{Y}}_n + \mathrm{diag} \, (\omega_{ns}^2 M_s) \boldsymbol{Y}_n - qb \sum_{m=0}^{N_0-1} \sum_{r=1}^{\infty} \Big[\Big(A_{nmr} \boldsymbol{Y}_{mr} + \frac{1}{\omega_r} B_{nmr} \dot{\boldsymbol{Y}}_{mr} \Big)$$

$$+ \varepsilon_1 \sum_{l=1}^{N_2} \Big(C_{nmrl}(t) \boldsymbol{Y}_{mr} + \frac{1}{\omega_r} D_{nmrl}(t) \dot{\boldsymbol{Y}}_{mr} \Big) \Big] = \varepsilon_1 qb \boldsymbol{F}_n \tag{4.1}$$

$$(n = 0, 1, 2, \ldots, N_0 - 1),$$

where $\boldsymbol{Y}_n = \sum\limits_{r=1}^{\infty} \boldsymbol{Y}_{nr}$, M_s is the generalized mass corresponding to the sth mode of vibrations, and ω_{ns} is the eigenfrequency of s-mode vibrations of the nth blade in vacuum. The presence of terms with time-dependent coefficients in this system shows that a parametric resonance can arise (vibrations become unstable owing to a certain combination of parameters). As was stated in [14], a necessary condition for the parametric resonance emergence is satisfaction of one of the following relations:

$$\omega_{0s} = n_1\omega_r(1+\delta)/2, \qquad |\delta| \leqslant \delta_{n_1 s} \qquad (n_1 = 1, 2, 3, \ldots) \tag{4.2}$$

(ω_{0s} is the eigenfrequency of s-mode vibrations of the cascade of blades in the flow; $\delta_{n_1 s}$ are small quantities whose values determine the boundaries of the domain of stability of parametric vibrations).

5. Forced Vibrations of the Cascade. The forced vibrations of the cascade, which are induced by free-stream nonuniformity, are described by a particular solution of the inhomogeneous system (4.1). If condition (4.2) in this system is not satisfied, then the terms in the right side containing time-dependent coefficients can be neglected. Moreover, as the aerodynamic forces are small as compared with elastic and inertial forces acting on the blades in the gas flow, we can introduce a small parameter $\varepsilon_0 = qb/(M\omega_{0s}^2)$, where M is the blade mass. Then, system (4.1) transforms to

$$\ddot{\boldsymbol{Y}}_n + \operatorname{diag}(\omega_{ns}^2)\boldsymbol{Y}_n = \varepsilon_0\omega_{0s}^2 \operatorname{diag}\Big(\frac{M}{M_s}\Big)\Big[\sum_{m=0}^{N_0-1}\sum_{r=1}^{\infty}\Big(A_{nmr}\boldsymbol{Y}_{mr} + \frac{1}{\omega_r}B_{nmr}\dot{\boldsymbol{Y}}_{mr}\Big) + \varepsilon_1\boldsymbol{F}_n\Big] \tag{5.1}$$
$$(n = 0, 1, 2, \ldots, N_0 - 1).$$

Substituting Eqs. (3.3) and (3.4) into Eq. (5.1) and equating separately the coefficients at $\sin(\omega_r t)$ and $\cos(\omega_r t)$ in the left and right sides of the corresponding equalities, we obtain a system of matrix equations for each time harmonic:

$$\operatorname{diag}\Big(1 - \frac{\omega_{ns}^2}{\omega_r^2}\Big)\boldsymbol{Y}'_{nr} = -\frac{\varepsilon_0}{r^2}\operatorname{diag}\Big(\frac{M}{M_s}\Big)\Big[\sum_{m=0}^{N_0-1}(A_{nmr}\boldsymbol{Y}'_{mr} + B_{nmr}\boldsymbol{Y}''_{mr}) + \varepsilon_1\operatorname{Re}(\boldsymbol{F}_{0r}\,\mathrm{e}^{j\mu_r n})\Big],$$
$$\operatorname{diag}\Big(1 - \frac{\omega_{ns}^2}{\omega_r^2}\Big)\boldsymbol{Y}''_{nr} = -\frac{\varepsilon_0}{r^2}\operatorname{diag}\Big(\frac{M}{M_s}\Big)\Big[\sum_{m=0}^{N_0-1}(A_{nmr}\boldsymbol{Y}''_{mr} - B_{nmr}\boldsymbol{Y}'_{mr}) + \varepsilon_1\operatorname{Im}(\boldsymbol{F}_{0r}\,\mathrm{e}^{j\mu_r n})\Big] \tag{5.2}$$
$$(n = 0, 1, 2, \ldots, N_0 - 1).$$

Here, $\boldsymbol{Y}'_{nr}$ and $\boldsymbol{Y}''_{nr}$ are the vectors whose components are the values of $y'_{nrs} = \alpha_{nrs}\cos\beta_{nrs}$ and $y''_{nrs} = -\alpha_{nrs}\sin\beta_{nrs}$, respectively.

6. Parametric Vibrations of the Cascade of Blades. The parametric vibrations of the cascade, which are induced by free-stream nonuniformity, are described by a homogeneous system of differential equations corresponding to Eq. (4.1). Following [14], the solution of this system is sought in the form

$$\boldsymbol{Y}_n = \mathrm{e}^{ht}\Big[\frac{1}{2}\boldsymbol{Y}'_{n0} + \sum_{r=1}^{\infty}(\boldsymbol{Y}'_{nr}\cos(r\omega t) + \boldsymbol{Y}''_{nr}\sin(r\omega t))\Big],$$

where h is the characteristic index of the solution and $\boldsymbol{Y}'_{rn}$ and $\boldsymbol{Y}''_{rn}$ are the vectors similar to the vectors in Eq. (5.2).

For conservative systems, we have $\operatorname{Re}(h) > 0$ under condition (4.2), i.e., vibrations become unstable. This condition, however, is insufficient for the loss of stability in systems with decaying processes, which is the case in the system considered owing to aerodynamic damping. The boundary of the region of stability of parametric vibrations is determined from the condition of balance between the energy of parametric excitation of vibrations and the energy spent on overcoming aerodynamic damping. In particular, if the free-stream nonuniformity in operation regimes of the cascade is low, then the probability of the parametric resonance emergence is also low, because the time-dependent coefficients of system (4.1) are small with respect to the coefficients responsible for aerodynamic damping. In regimes where the velocity of the flow through the cascade is close to the critical velocity of the classical flutter $\boldsymbol{W}_*$, which is determined in a uniform incoming flow, aerodynamic damping tends to zero. Therefore, in a certain vicinity of the mean velocity of the nonuniform incoming flow $\boldsymbol{W}_{01} = \boldsymbol{W}_* - \Delta\boldsymbol{W}$, where aerodynamic damping has still sufficiently low intensity, condition (4.2) can turn out to be sufficient for the emergence of instability of cascade vibrations.

It should be noted that, if this regime corresponds to the resonance region, i.e.,

$$|1-\omega_{0s}^2/\omega_r^2|<\varepsilon_0, \qquad \omega_r=\omega r, \tag{6.1}$$

then the amplitudes of forced vibrations of the blades are sufficiently high. In this case, there is no need to study their vibrations in more detail, because the operation of the corresponding structure in these regimes is inadmissible. Therefore, the parametric resonance phenomenon described by condition (4.2) at even values of n_1 can be eliminated from considerations. If condition (6.1) is not satisfied, Eq. (5.2) yields the estimate $\alpha_{nrs}=O(\varepsilon_0\varepsilon_1)$, which proves the statement given in Sec. 3.

It is known that the most dangerous region of instability of parametric vibrations is located in the vicinity of the values

$$\omega_r=2\omega_{0s}. \tag{6.2}$$

Following [14], in determining the boundaries of this region in the case of small values of the time-dependent coefficients, we can present the sought solution in the first approximation in the form

$$\boldsymbol{Y}_n=\boldsymbol{Y}'_{n1}\cos{(\omega t/2)}+\boldsymbol{Y}''_{n1}\sin{(\omega t/2)}. \tag{6.3}$$

Substituting Eq. (6.2) with allowance for Eq. (4.2) to the homogeneous system of differential equations corresponding to Eq. (4.1) and equating the coefficients at $\sin{(\omega t/2)}$ and $\cos{(\omega t/2)}$, we obtain an algebraic system of matrix equations of the form

$$\begin{gathered}
-(2\delta_{1s}+\delta_{1s}^2)\boldsymbol{Y}'_{n1}+\varepsilon_0\,\mathrm{diag}\Big(\frac{M}{M_s}\Big)\sum_{m=0}^{N_0-1}\Big[(A_{nm1}+\varepsilon_1G_{nm11})\boldsymbol{Y}'_{m1}+(B_{nm1}+\varepsilon_1H_{nm11})\boldsymbol{Y}''_{m1}\Big]=0,\\
-(2\delta_{1s}+\delta_{1s}^2)\boldsymbol{Y}''_{n1}+\varepsilon_0\,\mathrm{diag}\Big(\frac{M}{M_s}\Big)\sum_{m=0}^{N_0-1}\Big[(A_{nm1}-\varepsilon_1G_{nm11})\boldsymbol{Y}''_{m1}-(B_{nm1}-\varepsilon_1H_{nm11})\boldsymbol{Y}'_{m1}\Big]=0\\
(n=0,1,2,\ldots,N_0-1),
\end{gathered} \tag{6.4}$$

where G_{nm11} and H_{nm11} are the matrices whose elements are determined in the following manner with allowance for Eqs. (1.6) and (2.3):

$$g_{su}^{nml}=\int\limits_{R_0}^{R_{end}}\oint\limits_{L_n(z)}\mathrm{Re}\,(\bar{\boldsymbol{W}}_{11}\boldsymbol{w}_{mu1})\psi_s\,d\sigma\,dz, \qquad h^{nmll}=\int\limits_{R_0}^{R_{end}}\oint\limits_{L_n(z)}\mathrm{Im}\,(\bar{\boldsymbol{W}}_{11}\boldsymbol{w}_{mu1})\psi_s\,d\sigma\,dz.$$

The condition of existence of the solution of system (6.4) in the form (6.3) is the equation of the critical frequencies, which is obtained in the case of a zero determinant of the matrix composed of the coefficients at the unknown vectors $\boldsymbol{Y}'_{n1}$ and $\boldsymbol{Y}''_{n1}$ in this system. In addition to the sought quantity δ_{1s}, another unknown is the Strouhal number $k_{01}=\omega b/|\boldsymbol{W}_{01}|$, which affects the coefficients of aerodynamic forces. This number involves the mean free-stream velocity $\boldsymbol{W}_{01}$ on the boundaries of stability of parametric vibrations. The value of this velocity is of practical interest in solving problems of flutter of blade cascades. In accordance with the considerations presented above, the values of k_{01} should be sought in the vicinity of $k_*=\omega b/|\boldsymbol{W}_*|$ from the condition of balance between the forces exciting parametric vibrations of the blades and the forces of aerodynamic damping. The existence of a value $k_{01}>k_*$ satisfying this condition would mean that the critical flutter velocity could decrease by the following value under conditions (4.1) owing to free-stream nonuniformity:

$$|\Delta\boldsymbol{W}|=\omega b(1/k_*-1/k_{01}). \tag{6.5}$$

7. Reduction of the Critical Flutter Velocity of the Cascade, Caused by Free-Stream Nonuniformity. As an example, let us consider a nonuniform flow of an ideal incompressible fluid through a plane uniform cascade of blades with a period $T=2\pi/\omega$. Let the blades be able to perform vibrations with one degree of freedom, with the eigenfrequency of these vibrations being ω_{01}. Taking into account Eq. (6.3), we seek for the law of blade vibrations on the boundary of stability of parametric vibrations in the form

$$z_n=\alpha_{01r}\exp j(\omega t/2+\mu_rn), \qquad \mu_r=2\pi r/N_0 \qquad (r=0,1,2,\ldots,N_0-1). \tag{7.1}$$

Then, the expressions for the quantities y'_{n1r} and y''_{n1r} introduced in Eq. (5.2) become

$$y'_{n1r} = \alpha_{01r} \cos(n\mu_r), \qquad y''_{n1r} = -\alpha_{01r} \sin(n\mu_r). \tag{7.2}$$

Substituting Eq. (7.2) into Eq. (6.4) and taking into account Eqs. (4.2), (7.1), and (6.2), we obtain

$$\begin{gathered} \Big(\delta_{11r}(2 + \delta_{11r}) - \varepsilon_0 \frac{M}{M_1} \operatorname{Re}(c_0^r + d_0^r)\Big)\alpha_{01r} = 0, \\ \operatorname{Im}(c_0^r - d_0^r)\alpha_{01r} = 0, \end{gathered} \tag{7.3}$$

where c_0^r and d_0^r are the complex coefficients of the generalized forces acting on the initial blade ($n = 0$) in the case of simultaneous vibrations of the blades with identical amplitudes and a constant phase shift:

$$c_0^r = \sum_{m=0}^{N_0-1} c_{0m} \exp(j\mu_r m), \qquad d_0^r = \varepsilon_1 \sum_{m=0}^{N_0-1} d_{0m} \exp(j\mu_r m),$$

$c_{0m} = a_{11}^{0m1} + jb_{11}^{0m1}$ are the aerodynamic influence coefficients of vibrations of the mth blade in the uniform incoming flow on the magnitude of the generalized force acting on the initial blade; $d_{0m} = \varepsilon_1(g_{11}^{0m1} + jh_{11}^{0m1})$ are the corresponding coefficients due to flow nonuniformity.

For a nontrivial solution of the form (6.3) to exist, the coefficients at α_{01r} in Eqs. (7.3) should be equal to zero. In the first equation in (7.3), the zero coefficient at α_{01r} yields the equation of the critical frequencies. In the second equation, the zero coefficient means the condition of balance between the aerodynamic forces exciting parametric vibrations of the blades and the forces of aerodynamic damping.

It should be noted that the equation of the critical frequencies is valid for all values of the Strouhal number, which affects the aerodynamic influence coefficients. The Strouhal number that may cause the parametric resonance to emerge is found from the following condition with allowance for Eq. (7.3):

$$\operatorname{Im}(c_0^r - d_0^r) = 0. \tag{7.4}$$

To determine k_{01}, in accordance with Eq. (6.5), we introduce the quantity $\Delta k = k_{01} - k_*$ and find the approximate dependences of c_0^r and d_0^r on the Strouhal number in the vicinity of k_* in the form

$$c_0^r = c_{01}^r \Delta k_r, \qquad d_0^r = \varepsilon_1(d_{00}^r + d_{01}^r \Delta k_r).$$

Substituting these expressions into Eq. (7.4), we obtain

$$\Delta k_r = \frac{\varepsilon_1 d_{00}^r}{c_{01}^1 - \varepsilon_1 d_{01}^r}. \tag{7.5}$$

With allowance for Eqs. (6.5) and (7.5), the reduction of the critical flutter velocity due to flow nonuniformity can be approximately determined by the formula

$$\eta_r = \frac{|\Delta \boldsymbol{W}_r|}{|\boldsymbol{W}_*|} = \frac{\varepsilon_1 d_{00}^r}{k_* c_{01}^r + \varepsilon_1(d_{00}^r - k_* d_{01}^r)}. \tag{7.6}$$

Let us determine the critical flutter velocity of the cascade (its solidity $\tau = b/h = 0.5$, angle of mounting $\beta = -30^\circ$, and parameter of blade bending $\bar{f} = 0.1$) as a function of flow nonuniformity in the case of twisting vibrations of the blades, using the aerodynamic influence coefficients calculated in [5]. In the case of blade vibrations with the phase shift $\mu_r = \pi/2$, with the most adverse initial data determining the phase of blade vibrations with respect to the phase of the first harmonic of free-stream nonuniformity, we obtain the following values of the coefficients in Eq. (7.6): $k_* = 0.265$, $c_{01}^r = 2.4$, $d_{00}^r = 0.9$, and $d_{01}^r = -0.62$. Substituting these values into Eq. (7.6), we obtain

$$\eta_r = \frac{0.9\varepsilon_1}{0.636 + 1.064\varepsilon_1}.$$

Thus, we can conclude that the nonuniformity of the velocity field of the flow through the cascade can exert a significant effect on the critical flutter velocity of the cascade blades.

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